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DETERMINATION OF THE EFFECTIVE ELASTIC MODULI OF INHOMOGENEOUS MATERIALS

V. V. Novikov

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1. FORMULATION OF THE PROBLEM

Quasihomogeneous media that possess effective properties dependent on the properties, volume concentration, and contact conditions of the components are usually investigated when examining the effective properties of inhomogeneous materials. The necessary and sufficient condition for going over to the quasihomogeneous medium is compliance of the dimension of the inhomogeneity l with the inequality

$$l_0 \ll l \ll L, \tag{1.1}$$

where l_0 is the crystal lattice constant and L is the specimen dimension.

The effective elastic moduli C_{ijkl} and the pliability S_{ijkl} are determined from the equations

$$\langle\sigma_{ij}\rangle = C_{ijkl}\,\langle\varepsilon_{kl}\rangle,\ \langle\varepsilon_{ij}\rangle = S_{ijkl}\ \langle\sigma_{kl}\rangle. \tag{1.2}$$

The angular brackets <...> here denote taking the average over the volume of the material

$$\langle\sigma_{ij}\rangle = \frac{1}{V}\iiint\limits_V \sigma_{ij}(\mathbf{r})\,dx_1dx_2dx_3,\quad \langle\varepsilon_{ij}\rangle = \frac{1}{V}\iiint\limits_V \varepsilon_{ij}(\mathbf{r})\,dr_1dr_2dr_3. \tag{1.3}$$

The equations

$$\sigma_{ij}(\mathbf{r}) = C_{ijkl}(\mathbf{r})\varepsilon_{kl}(\mathbf{r}),\ \varepsilon_{ij}(\mathbf{r}) = S_{ijkl}(\mathbf{r})\sigma_{kl}(\mathbf{r}), \tag{1.4}$$

are valid for the local domains (components) when conditions (1.1) are satisfied, where $\sigma_{ij}(\mathbf{r})$ is the local stress tensor, $\varepsilon_{ij}(\mathbf{r})$ is the local strain tensor, and $\mathbf{r} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ is a radius-vector.

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In the general case, determination of the effective elastic moduli C_{ijkl} reduces to solving the equations [1]

$$C_{ijkl} = C^{(1)}_{ijmn} v_1 A^{(1)}_{mnkl} + C^{(2)}_{ijmn} v_2 A^{(2)}_{mnkl},$$
$$v_1 A^{(1)}_{mnkl} + v_2 A^{(2)}_{mnkl} = \frac{1}{2}(\delta_{mk}\delta_{nl} + \delta_{ml}\delta_{nk}), \quad (1.5)$$

where δ_{kl} is the Kronecker delta, and $A^{(i)}_{mnkl}$ are unknown tensors determined from the equations

$$\langle \varepsilon^{(1)}_{kl} \rangle = A^{(1)}_{klmn} \langle \varepsilon_{mn} \rangle, \quad \langle \varepsilon^{(2)}_{kl} \rangle = A^{(2)}_{klmn} \langle \varepsilon_{mn} \rangle,$$

where v_i is the bulk concentration of the i-th component, i = 1, 2.

The superscripts at the tensors and the subscripts at the scalars indicate to which component the given quantity refers.

Additional information is needed to determine C_{ijkl} from (1.5), since three unknowns (C_{ijkl}, $A^{(1)}_{klmn}$, $A^{(2)}_{klmn}$) are in the system (1.5) and there are two equations.

Information about the structure of the composite [2] can be that needed to close the system (1.5).

In the general case, the problem of closing (1.5) for a chaotic distribution of components in an inhomogeneous medium is analogous to the many particle problem in the theory of fluids [3]. The mathematical difficulties that occur in closing (1.5) would result in the appearance of several approximate methods of determining the effective elastic moduli of the composites: a variational method [4], a statistical theory of elasticity and a method of random functions [5-8], and a self-consistent field method [9, 10]. These methods are surveyed, for example, in [8, 11-13].

Formulas are determined below for bilateral estimates of the elastic moduli which permit taking account of the specific structure of an inhomogeneous material. The method of step-by-step quasihomogenization is used in determining the effective elastic characteristics together with the geometric simulation of the structure of the inhomogeneous material [14]. The crux of this method is the following: A representative volume V of the inhomogeneous material is first isolated, and the volume V is then divided into domains and the effective properties of these partition domains are determined; by considering the partition domains quasihomogeneous with known effective properties, we determine the effective properties of the whole representative element.

2. ELASTIC MODULI

Let the operation of taking the average of an arbitrary function f(r) with respect to the coordinates x_1, x_2, x_3 be

The mean with respect to the section x_k = const of the volume V whose area equals $S(x_k)$

$$\{f(\mathbf{r})\}_{S(x_k)} = \frac{1}{S(x_k)} \iint_{(D)} f(\mathbf{r})\, dx_i dx_j,$$

where D is the projection of $S(x_k)$ on the coordinate plane Ox_ix_j;

The mean of the line $L(x_i, x_j)$ passing through a point with coordinates (x_i, x_j) parallel to the Ox_k axis, with respect to the length

$$\{f(\mathbf{r})\}_{L(x_i,x_j)} = \frac{1}{L(x_i, x_j)} \int_0^L f(\mathbf{r})\, dx_k.$$

The strain potential energy of the body per unit volume V is

$$U = (1/2V) \langle \varepsilon_{ij}(\mathbf{r}) \sigma_{ij}(\mathbf{r}) \rangle.$$

For a quasihomogeneous medium U can be written in the form $U = (1/2)\langle\varepsilon_{ij}\rangle\langle\sigma_{ij}\rangle$. Here the relationships (1.2) are valid for $\langle\sigma_{ij}\rangle$ and $\langle\varepsilon_{ij}\rangle$.

It follows from the condition of minimum potential energy that for any trial functions $\sigma'_{ij}(\mathbf{r})$ and $\varepsilon'_{ij}(\mathbf{r})$ satisfying the same boundary conditions as $\sigma_{ij}(\mathbf{r})$ and $\varepsilon_{ij}(\mathbf{r})$, the following will be satisfied

$$U' \geqslant U, \quad U' = \frac{1}{2V}\langle \varepsilon'_{ij}(\mathbf{r})\rangle \langle \sigma'_{ij}(\mathbf{r})\rangle. \tag{2.1}$$

Let us examine two methods of selecting the trial functions $\sigma'_{ij}(\mathbf{r})$ and $\varepsilon'_{ij}(\mathbf{r})$, which permit determination of the upper and lower bounds of the effective elastic moduli C_{ijkl} on the basis of the inequalities (2.1).

First Method. We select the trial function $\sigma'_{ij}(\mathbf{r})$ in such a way that

$$\frac{\partial}{\partial x_k}\{\sigma'_{ij}(\mathbf{r})\}_S = 0, \tag{2.2}$$

is satisfied, i.e., the stress tensor $\sigma'_{ij}(\mathbf{r})$ averaged over the section is independent of the coordinates x_k. Here S is the area of the projection of the representative volume V on the plane Ox_ix_j.

If it is taken into account that in the general case

$$\{\sigma'_{ij}(\mathbf{r})\}_S = \{C_{ijkl}(\mathbf{r})\,\varepsilon_{kl}(\mathbf{r})\}_S,$$

is satisfied, then we can write

$$\{\sigma'_{ij}(\mathbf{r})\}_S = H_{ijkl}(x_k)\{\varepsilon_{kl}(\mathbf{r})\}_S, \tag{2.3}$$

where $H_{ijkl}(x_k)$ is the tensor of the elastic modulus of a layer of thickness dx_k perpendicular to the Ox_k axis

$$H_{ijkl}(x_k) = C^{(2)}_{ijkl} I_{mnkl} + \bar{S}_1(x_k)\left(C^{(1)}_{ijkl} - C^{(2)}_{ijkl}\right) A^{(1)}_{mnkl}(x_k), \tag{2.4}$$

where $S_1(x_k)$ is the area of a section through the representative volume V by the plane x_k = const occupied by the first component; $\bar{S}_1(x_k) = S_1(x_k)/S(x_k)$; $S(x_k) = S_1(x_k) + S_2(x_k)$; $I_{mnkl} = (1/2)(\delta_{mk}\delta_{nl} + \delta_{ml}\delta_{nk})$.

The tensor $A^{(1)}_{mnkl}(x_k)$ is determined from the relationship

$$\{\varepsilon^{(1)}_{kl}(\mathbf{r})\}_{S_1} = A^{(1)}_{klmn}(x_k)\,\{\varepsilon_{mn}(\mathbf{r})\}_S.$$

Multiplying (2.3) by the reciprocal tensor $[H_{ijkl}(x_k)]^{-1}$ and then taking the average with respect to the variable x_k with (2.2) taken into account, we obtain

$$\langle \varepsilon_{kl}\rangle = \{[H_{ijkl}(x_k)]^{-1}\}_L \langle \sigma'_{ij}\rangle, \tag{2.5}$$

where L is the length of the projection of the representative volume V along the Ox_k axis.

According to the inequality (2.1), we determine the upper bound for C_{ijkl}

$$[\{[H_{ijkl}(x_k)]^{-1}\}_L]^{-1} \leqslant C_{ijkl}. \tag{2.6}$$

Second Method. We now select the trial function $\varepsilon'(\mathbf{r})$ in such a manner as to satisfy

$$\frac{\partial}{\partial x_l}\{\varepsilon'_{ij}(\mathbf{r})\}_L = 0, \tag{2.7}$$

i.e., the deformation of a prism of length L with base area dx_idx_j is constant (is independent of the coordinates x_i, x_j).

Taking into account that

$$\{\varepsilon'_{ij}(\mathbf{r})\}_L = \{S_{ijkl}(\mathbf{r})\,\sigma_{kl}(\mathbf{r})\}_L,$$

we write on the basis of the linearity of the problem

$$\{\varepsilon'_{ij}(\mathbf{r})\}_L = M_{ijkl}(x_i, x_j)\{\sigma_{kl}(\mathbf{r})\}_L, \tag{2.8}$$

$M_{ijkl}(x_i, x_j)$ is the pliability tensor of a prism of length L with base area dx_idx_j equal to

$$M_{ijkl}(x_i, x_j) = S^{(2)}_{ijkl} I_{mnkl} + \bar{L}_1(x_i, x_j)\left(S^{(1)}_{ijkl} - S^{(2)}_{ijkl}\right) B^{(1)}_{mnkl}(x_i, x_j), \tag{2.9}$$

where $L_1(x_i, x_j)$ is the length of a line passing parallel to the Ox_k through the representative volume V along the first components $\bar{L}_1(x_i, x_j) = L_1(x_i, x_j)/L(x_i, x_j)$; $L(x, x_j) = L_1(x_i, x_j) + L_2(x_i, x_j)$.

The tensor $B^{(1)}_{mnkl}(x_i, x_j)$ is determined from the equality

$$\{\sigma^{(1)}_{mn}(\mathbf{r})\}_{L_1} = B^{(1)}_{mnkl}(x_i, x_j)\{\sigma_{kl}(\mathbf{r})\}_L.$$

Formulas (2.3) and (2.8) are written under the assumption of linearity of the elasticity equations.

Multiplying (2.8) by the reciprocal tensor $[M_{ijkl}(x_i, x_j)]^{-1}$ and then integrating with respect to the variables x_i, x_j with (2.7) taken into account, we obtain

$$\langle\sigma_{kl}\rangle = \{[M_{ijkl}(x_i, x_j)]^{-1}\}_S\langle\varepsilon'_{ij}\rangle. \tag{2.10}$$

On the basis of the inequality (2.1) with (2.10) taken into account, we determine the lower bound for C_{klij} in the form

$$C_{ijkl} \geqslant \{[M_{ijkl}(x_i, x_j)]^{-1}\}_S. \tag{2.11}$$

Combining the inequalities (2.6) and (2.11), we have

$$\{[M_{ijkl}(x_i, x_j)]^{-1}\}_S \leqslant C_{ijkl} \leqslant [\{[H_{ijkl}(x_k)]^{-1}\}_L]^{-1}. \tag{2.12}$$

If the components of the inhomogeneous material are isotropic and homogeneous, then the elastic modulus tensor C_{ijkl} and the pliability tensor S_{ijkl} can be represented as the sum of the volume and deviator components

$$C_{ijkl} = 3KV_{ijkl} + 2\mu D_{ijkl},$$
$$S_{ijkl} = (1/3K)V_{ijkl} + (1/2\mu)D_{ijkl},$$

where V_{ijkl} and D_{ijkl} are the volume and deviator parts of a unit tensor of the fourth rank

$$V_{ijkl} = \frac{1}{3}\delta_{ij}\delta_{kl}, \quad D_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right)$$

(K is the bulk, and μ the shear elastic modulus).

Since the potential energy of an elastic body can be represented in the form of the sum of the multilateral compression potential energy and the pure shear potential energy, then the inequalities (2.12) will be valid separately for the volume and deviator parts of the elastic moduli tensors

$$\{K(x_i, x_j)\}_S \leqslant K \leqslant [\{[K(x_k)]^{-1}\}_L]^{-1}; \tag{2.13}$$
$$\{\mu(x_i, x_j)\}_S \leqslant \mu \leqslant [\{[\mu(x_k)]^{-1}\}_L]^{-1}, \tag{2.14}$$

where $K(x_i, x_j)$, $\mu(x_i, x_j)$ are the bulk and shear moduli, respectively, determined from (2.9), and $K(x_k)$ and $\mu(x_k)$ are the bulk and shear moduli determined from (2.4).

Taking account of the assumptions made (2.2) and (2.7), the expressions for $K(x_i, x_j)$, $K(x_k)$ and $\mu(x_i, x_j)$, $\mu(x_k)$ can be obtained from (2.9) and (2.4) in the form

$$K(x_i, x_j) = \left(\left\{\frac{n}{K}\right\}_L - 2\frac{\{d\}_L\cdot\{P\}_L}{\{KP\}_L}\right)^{-1}, \quad \mu(x_i, x_j) = \left\{\frac{1}{\mu}\right\}_L^{-1}; \tag{2.15}$$
$$K(x_k) = \{KP\}_S/\{P\}_S, \ \mu(x_k) = \{\mu\}_S, \tag{2.16}$$

where

$$P_i = 6m_i/(3 + 4m_i);\ d_i = (3 - 2m_i)/(3 + 4m_i);\ n_i = 9/(3 + 4m_i);\ m_i = \mu_i/K_i;$$
$$\{f\}_L = \bar{L}_1(x_i, x_j)f_1 + \bar{L}_2(x_i, x_j)f_2;\ \{f\}_S = \bar{S}_1(x_k)f_1 + \bar{S}_2(x_k)f_2;$$

$S_i(x_k)$ is the area of the section of the volume V perpendicular to the Ox_k axis and occupied by the i-th component (i = 1, 2); $S(x_k) = S_1(x_k) + S_2(x_k)$; $\bar{S}_i(x_k) = S_i(x_k)/S(x_k)$; $L_i(x_i, x_j)$ is the length of the segment passing parallel to the Ox_k axis in the i-th component, and $L(x_i, x_j) = L_1(x_i, x_j) + L_2(x_i, x_j)$; $\bar{L}_i(x_i, x_j) = L_i(x_i, x_j)/L(x_i, x_j)$.

If $\bar{S}_1(x_k) = v_1$, $\bar{S}_2(x_k) = v_2$, $\bar{L}_1(x_i, x_j) = v_1$, $\bar{L}_2(x_i, x_j) = v_2$, then taking account of (2.15) we obtain from (2.13)

$$\left(\left\langle\frac{n}{K}\right\rangle - 2\frac{\langle d\rangle\langle P\rangle}{\langle KP\rangle}\right)^{-1} \leqslant K \leqslant \frac{\langle KP\rangle}{\langle P\rangle}; \tag{2.17}$$
$$\langle 1/\mu\rangle^{-1} \leqslant \mu \leqslant \langle\mu\rangle. \tag{2.18}$$

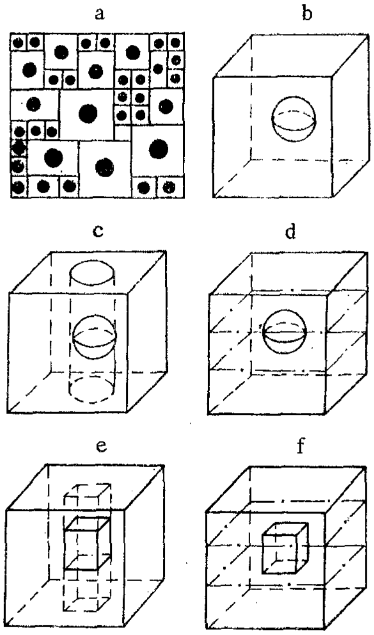

Fig. 1

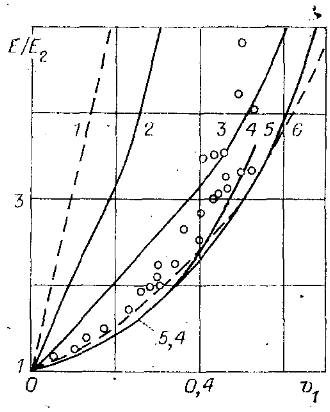

Fig. 2

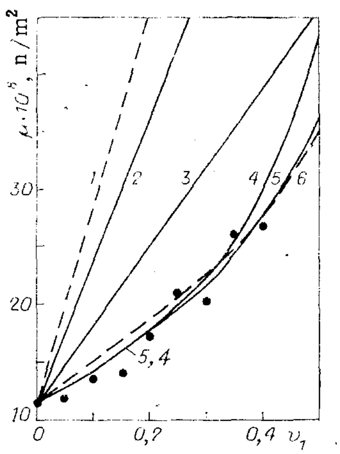

Fig. 3

When the Poisson ratios of the mixture components are equal, then (2.17) takes the form

$$\langle 1/K\rangle^{-1} \leqslant K \leqslant \langle K\rangle. \tag{2.19}$$

In this case (2.18) and (2.19) agree with the Voigt and Royce bracket for K and μ [8].

It must be noted that the difference between the bracket obtained for the elastic moduli (2.12) and an analogous estimate of the Hashin–Shtrikman [4] elastic moduli boundaries is that the formulas in the inequality (2.12) permit taking into account the microstructure of the inhomogeneous material in greater detail, and therefore, closing the bracket for K and μ. In our case, the microinhomogeneities can here have arbitrary form, possess anisotropic properties, and are randomly arranged in the volume, which also enlarges the possibility of the application of the inequality (2.12) as compared with the bilateral estimates obtained under the assumption of strict periodicity in the location of inclusions of normal form and isotropy of the components [15]. As an illustration of utilization of the formulas obtained, we consider a structure with isolated inclusions.

3. SPHERE IN A CUBE

Let us determine the upper and lower bounds for the moduli K and μ by using (2.13) and (2.14) for a macroscopically homogeneous and isotropic material consisting of a homogeneous and isotropic matrix and inclusions of spherical shape of the other component.

We shall consider that each inclusion is surrounded by a surface S_n lying entirely in the matrix and a bounding volume V_n such that $V_1/V_n = v_1$, where V_1 is the volume of the inclusion (Fig. 1a). It is assumed that the volume V_n has the shape of cubes of all sizes (from finite to infinitesimal) so that they can fill the whole volume of the material (Fig. 1b).

Lower Bound. We divide the elementary cell of the sphere in the cube into two domains: we isolate a cylinder whose generators are parallel to the Ox_3 axis while the radius equals the radius of the sphere R (Fig. 1c). In this case

$$\bar{L}_1(x_j, x_j) = \pi_1\sqrt{1-\rho^2},\ \bar{L}_2(x_1, x_2) = 1 - \bar{L}_1(x_j, x_j), \tag{3.1}$$

where

$$\pi_1 = 2(3v_1/4\pi)^{1/3},\quad \rho^2 = \bar{x}_1^2 + \bar{x}_2^2,\quad \bar{x}_i = x_i/R.$$

Substituting (3.1) into (2.15) and then using the expression for the lower bound in inequalities (2.13) and (2.14), we determine the properties of the cylinder first (the bulk and the shear γ_i' moduli), and then the effective properties of the elementary cell

$$\varkappa_i' = 2\frac{C_2}{\pi_1^2}\ln\frac{P_2}{A_3\pi_1^2 + A_4\pi_1 + P_2} - \frac{C_1}{\pi_1} + \frac{C_3}{\pi_1^3}\left[I_{C_4}(C_5) - I_{C_4}(1 - C_5)\right]; \tag{3.2}$$

$$\gamma' = \frac{2\mu_2}{(t-1)\pi_1}\left\{1 - \frac{1}{(t-1)\pi_1}\ln\left[(t-1)\pi_1 + 1\right]\right\}, \tag{3.3}$$

where

$$A_1 = K_1P_1 - K_2P_2, \quad A_2 = K_2P_2, \quad A_3 = P_1 + P_2 - A, \quad t = \frac{\mu_2}{\mu_1},$$
$$A = \left(\frac{K_2}{K_1}\pi_1 - 2d_1\right)P_2 + \left(\frac{K_1}{K_2}\pi_1 - 2d_2\right)P_1, \quad A_4 = A - 2P_2,$$
$$C_1 = A_1/A_3, \quad C_2 = (A_2A_3 - A_4A_1)/(2A_3^2), \quad C_3 = \left(A_4^2A_1 - A_2A_3A_4 - \right.$$
$$\left. - 2A_3A_5A_1\right)/(4A_3^3), \quad C_4 = (A_2 - 4P_1P_2)/(4A_3\pi_1^2), \quad C_5 = \frac{A_4}{2A_3\pi_1}. \tag{3.4}$$

Here

$$I_b(z) = \begin{cases} \dfrac{1}{2\sqrt{b}}\ln\dfrac{z-\sqrt{b}}{z+\sqrt{b}}, & b > 0, \\ \dfrac{1}{z}, & b = 0, \\ \dfrac{1}{\sqrt{|b|}}\operatorname{arctg}\dfrac{z}{\sqrt{|b|}}, & b < 0. \end{cases} \tag{3.5}$$

We finally obtain the lower bound for the moduli K' and μ' in the form

$$K' = \frac{K_2P_2 + (\varkappa'P' - K_2P_2)\pi_2}{P_2 + (P' - P_2)\pi_2}; \tag{3.6}$$

$$\mu' = \mu_2 + (\gamma' - \mu_2)\pi_2, \quad m' = \frac{\gamma'}{\varkappa'}, \quad \pi_2 = \pi^{1/3}\left(\frac{3v_1}{4}\right)^{2/3}. \tag{3.7}$$

Upper Bound. In this case we divide the elementary cell into two domains as follows: We draw tangent planes to the sphere perpendicular to the Ox_3 axis (Fig. 1d). For the domain between the tangential planes, we introduce the notation: $\varkappa''$ is the bulk modulus, and γ'' the shear modulus. Here

$$\bar{S}_1(x_3) = \pi_2\left(1 - \bar{x}_3^2\right), \quad \bar{S}_2(x_3) = 1 - \bar{S}_1(x_3), \quad \bar{x}_3 = x_3/R. \tag{3.8}$$

Substituting (3.8) into (2.16), and then using the expressions for the upper boundaries of inequalities (2.13) and (2.14), we have

$$\varkappa'' = \left[\frac{B_2}{B_1} - \frac{B_1B_4 - B_2B_3}{B_4^2} I_{B_5}(1)\right]^{-1}; \tag{3.9}$$

$$\gamma'' = \mu_2(1-t)\left[I_{B_6}(1)\right]^{-1}; \tag{3.10}$$

$$B_1 = P_2 + B_2, \; B_2 = (P_1 - P_2)\pi_2, \; B_6 = \pi_2 + t/(1-t); \tag{3.11}$$

$$B_3 = K_2P_2 + B_4, \; B_4 = \pi_2(K_1P_1 - K_2P_2), \; B_5 = B_3/B_4. \tag{3.12}$$

Considering the domain between the tangent planes to the sphere as quasihomogeneous with effective properties $\varkappa''$ and γ'', we determine the effective moduli K'' and μ'' of the elementary cell

$$K'' = \left\{\frac{n_2}{K_2} + \left(\frac{n''}{\varkappa''} - \frac{n_2}{K_2}\right)\pi_1 - 2\frac{[d_2 + (d'' - d_2)\pi_1][P_2 + (P'' - P_2)\pi_1]}{K_2P_2 + (\varkappa''P'' - K_2P_2)\pi_1}\right\}^{-1}; \tag{3.13}$$

$$\mu'' = \left(\frac{1-\pi_1}{\mu_2} + \frac{\pi_1}{\gamma''}\right)^{-1}; \tag{3.14}$$

$$n'' = 9/(3 + 4m''), \; d'' = (3 - 4m'')/(3 + 4m''), \; P'' = 6m''/(3 + 4m''), \tag{3.15}$$

$$m'' = \gamma''/\varkappa''. \tag{3.16}$$

4. CUBE IN A CUBE

In the problem considered above we replace the sphere by a cube of the same volume. In this case the elementary cell will have the form displayed in Fig. 1e. For this cell all the calculations are simplified substantially.

Lower Bound. The moduli K' and μ' can be determined in the form

$$K' = \frac{K_2P_2 + (\varkappa'P' - K_2P_2)\alpha^2}{P_2 + (P' - P_2)\alpha^2}, \quad \alpha = v_1^{1/3}; \tag{4.1}$$

$$\mu' = \mu_2 + (\gamma' - \mu_2)\alpha^2, \tag{4.2}$$

where

$$\varkappa' = \left\{\frac{n_2}{K_2} + \left(\frac{n_1}{K_1} - \frac{n_2}{K_2}\right)\alpha - 2\,\frac{[d_2 + (d_1 - d_2)\alpha][P_2 + (P_1 - P_2)\alpha]}{P_2K_2 + (P_1K_1 - P_2K_2)\alpha}\right\}^{-1}; \tag{4.3}$$

$$\gamma' = \left(\frac{\alpha}{\mu_1} + \frac{1-\alpha}{\mu_2}\right)^{-1}. \tag{4.4}$$

Upper Bound. Partitioning the elementary cell into domains as indicated in Fig. 1f, the upper bound can be determined for the volume K'' and shear μ'' moduli in the form

$$K'' = \left\{\frac{n_2}{K_2} + \left(\frac{n''}{K''} - \frac{n_2}{K_2}\right)\alpha - 2\,\frac{[d_2 + (d'' - d_2)\alpha][P_2 + (P'' - P_2)\alpha]}{P_2K_2 + (\varkappa''P'' - K_2P_2)\alpha}\right\}^{-1}; \tag{4.5}$$

$$\mu'' = \left(\frac{1-\alpha}{\mu_2} + \frac{\alpha}{\gamma''}\right)^{-1}; \tag{4.6}$$

$$\varkappa'' = \frac{K_2P_2 + (K_1P_1 - K_2P_2)\alpha^2}{P_2 + (P_1 - P_2)\alpha}; \tag{4.7}$$

$$\gamma'' = \mu_2 + (\mu_1 - \mu_2)\alpha^2. \tag{4.8}$$

5. COMPARISON WITH EXPERIMENTAL DATA

The present paper is similar in approach to Hill, Hashin, and Shtrikman; hence, a computation by the Hashin–Shtrikman formulas is presented for a comparison between the formulas obtained and the experimental data. Experimental data in Figs. 2 and 3 are compared with a computation using (3.2)-(3.15) and (4.1)-(4.8). The experimental points are presented for an epoxy resin–quartz system [16, 17]. The volume concentration of the core in the system varies within the range $0 \leqslant v_1 \leqslant 0.5$.

Comparison shows that the bracket for the elastic moduli, computed on the basis of the model of spheres in a cube (curves 4 and 2) and cubes in a cube (curves 5 and 3) is narrower than the Hashin–Shtrikman bracket [4] (curves 6 and 1). Here the lower bounds for the Young's and shear moduli are practically in agreement for all three computation schemes in the range $0 \leqslant v_1 \leqslant 0.4$. Narrowest and sufficiently well encompassing the experimental data is the bracket for the elastic modulus obtained on the basis of the model of cubes in a cube. Consequently, this model and (4.1)-(4.8) can be recommended for computing systems of the continuous matrix-isolated inclusions type.

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VISCOPLASTIC DEFORMATION OF ANNULAR PLATES

S. N. Kosorukov UDC 539.374

Viscoplasticity is one of the most reliable and convenient methods of taking account of the dependence of the strength properties of materials on the loading rate [1, 2]. Analytic solutions of problems of quasistatic loading of sufficiently complex structure elements, which are convenient to obtain by linearizing the fundamental nonlinear viscoplasticity relationships, are of significant interest for practice.

This paper illustrates the utilization of one of the possible linearization methods. The solutions obtained for hinge-supported and clamped annular plates satisfy both the kinematic conditions and the equilibrium equations exactly.

1. A generalization of the simplest dependences for a stiffly viscoplastic material is presented in [1] and reduces to a dynamic flow criterion of the form

$$\sqrt{J_2} = k\left[1 + \Phi^{-1}\left(\frac{\sqrt{I_2}}{\gamma}\right)\right], \tag{1.1}$$

where k is the shear yield point, J_2, I_2 are the second invariants of the stress and strain rate deviators, γ is a coefficient characterizing the ratio between the viscous and plastic properties of the material, Φ is the symbol for a certain function, and Φ^{-1} is the symbol of the reciprocal function.

The associated flow law remains valid. The nonlinear Mises condition is used here as the initial flow condition in stresses. The radius of the circular cylindrical flow surface in the space of the principal stresses is determined also by a nonlinear combination of the principal strain rates. It is easy to see that points of the ellipse (Fig. 1) in the plane of the principal strain rates $\varepsilon_1 - \varepsilon_2$ correspond to points lying on an ellipse similar to the Mises ellipse in the plane of the principal stresses $\sigma_1 - \sigma_2$ for the plane stress state of an incompressible material. To linearize the initial nonlinear relationship it is sufficient to replace the ellipses by certain similar polygons by conserving the similarity of such polygons as the sizes change. For instance, if the ellipse J_2 = const is replaced by the hexagon 1 (Fig. 1a), similar to the Tresk hexagon, then by replacing the ellipse I_2 = const by hexagons 1 or 2 (Fig. 1b), we obtain the relationships, respectively, for the linear function F

$$\max(\sigma_\alpha - \sigma_\beta) = \sigma_T + \mu \max|\varepsilon_\gamma|, \quad \max(\sigma_\alpha - \sigma_\beta) = \sigma_T + (1/2)\mu|\varepsilon_\alpha - \varepsilon_\beta|, \tag{1.2}$$

where the subscripts α, β correspond to the maximal and minimal values of the quantities; ε_γ is the maximal strain rate in absolute value, and $\mu = 3k/2\gamma$ is the viscosity coefficient

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